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THE FORCED OSCILLATION OF SHALLOW DRAFT SHIPS

By W. D. Kim[1]

Considering a three-dimensional disk with a circular or elliptic waterplane as a ship of shallow draft, the interaction between the disk oscillating with six degrees of freedom and the induced motion of the surrounding fluid is investigated. The potential problem is formulated by the use of the concept of surface distributed sources so that an integral equation for the source density is obtained. The numerical solution of the integral equation is found and the dependence of added mass, added moment of inertia, and damping factors on various frequencies is determined.

I. INTRODUCTION

Consider the forced oscillation of a rigid disk on a free surface. The form of the disk is such that it will create large wave-making effects. The present analysis thus serves as a complement to the thin-ship theory [1, 3][2]. When the draft is small, the normal derivative of Green's function (the potential due to a unit source) does not enter into the kernel of the integral equation, and moreover the kernel can be evaluated explicitly. However, for arbitrary cross-sections [2], the presence of the normal derivative of Green's

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[2] Numbers in brackets designate References at end of paper.

function makes the numerical solution extremely difficult.

It is assumed that the amplitude of oscillation is sufficiently small so that the problem can be linearized. One may choose the ratio of actual draft to the wavelength ϵ as a small-parameter and develop a perturbation procedure following the method of MacCamy [3]. It then turns out that the added mass, added moment of inertia, and damping factors associated with surge, sway, and yaw are terms of order ϵ. Furthermore, this procedure reduces the potential problem of an oscillating disk to a series of boundary value problems of increasing orders in ϵ.

The solution of the lowest order problem is sought in the form of a potential due to the distribution of sources over the disk. Then, the singular property of surface distributed sources enables one to determine the density from an integral equation which satisfies the boundary values on the disk. The numerical solution of this integral equation can be obtained approximately by use of a finite set of linear equations which relate the unknown densities at selected pivotal points.

A scheme has been developed for evaluating the singular integral in two variables. (The fundamental singularities are log r and $\frac{1}{r}$.) From the solutions of the linear equations, pivotal values of pressure can be obtained. Thus, knowing the pressure distribution, one can evaluate added mass, added moment of inertia, and damping factors for heave, roll, and pitch corresponding to a specific frequency. It was found that accurate results for high frequency motion require a large number of pivotal points.

II. GENERAL FORMULATION

Suppose the plane $\bar{y} = 0$ represents the undisturbed free surface of an inviscid and incompressible fluid; $O\bar{x}$ and $O\bar{z}$ are axes which lie in that surface. The hull of a ship in the equilibrium position is given by

$$\bar{y} = S^{o}(\bar{x}, \bar{z}) \quad , \tag{2.1}$$

where the form of the ship is assumed to satisfy the following conditions:

$$\left.\begin{array}{lll} S^{o}(\bar{x}, \bar{z}) = 0 & \text{along the edge of the ship,} & \\ S^{o}(\bar{x}, \bar{z}) = \bar{\epsilon}, & \text{maximum draft at the center,} & \\ S^{o}(\bar{x}, \bar{z}) = S^{o}(-\bar{x}, \bar{z}), & \text{and} \quad S^{o}(\bar{x}, \bar{z}) = S^{o}(\bar{x}, -\bar{z}). & \end{array}\right\} \tag{2.2}$$

The last two conditions indicate that in the equilibrium position, the center of the waterplane coincides with the origin O of a time-independent reference frame which is called the space frame $O\bar{x},\bar{y},\bar{z}$.

Now, suppose the ship is excited into small oscillations. After transient disturbances have been dissipated, the resulting fluid motion will be periodic with a frequency of oscillation σ. Considering such a motion to be irrotational, the velocity potential may be written in the form

$$\phi(x,y,z;t) = \mathrm{Re}[U(x,y,z)e^{-i\sigma t}] \quad . \tag{2.3}$$

At any instant, the position of the oscillating ship may be described in terms of a position vector $\underline{R} = i\bar{X} + j\bar{Y} + k\bar{Z}$, and the Eulerian angles

$\underline{\theta} = i\theta_x + j\theta_y + k\theta_z$, where i, j, k denote unit vectors along axes of the space frame. One calls the components $\bar{X}$, $\bar{Y}$, and $\bar{Z}$, surge, heave and sway, respectively, and the angular components θ_x, θ_y, and θ_z, roll, yaw and pitch, respectively. Since the motion of the ship is periodic in time, one has

$$\left.\begin{aligned} \underline{R}(t) &= \mathrm{Re}[\underline{R}^o e^{-i\sigma t}] \quad , \\ \underline{\Theta}(t) &= \mathrm{Re}[\underline{\Theta}^o e^{-i\sigma t}] \quad , \end{aligned}\right\} \qquad (2.4)$$

where $\underline{R}^o$ and $\underline{\Theta}^o$ are amplitudes of the linear and angular displacements, respectively. From rigid body dynamics, the velocity of a particle on a ship is given by

$$\dot{\underline{r}} = \dot{\underline{R}} + \dot{\underline{\theta}} \times \underline{r}' \quad , \qquad (2.5)$$

where $\underline{r}$ is the position vector with respect to the space frame, while $\underline{r}'$ is the position vector with respect to a moving frame attached to the ship.

In order to show the physical significance of small oscillations, independent and dependent variables are made dimensionless by dividing each variable by the wave length. The space coordinates then become

$$x = k\bar{x} \ , \qquad y = k\bar{y} \ , \qquad z = k\bar{z} \ , \qquad (2.6)$$

where k represents the wave number which is equal to σ^2/g (or $2\pi/\bar{\lambda}$), g being the acceleration of gravity, and $\bar{\lambda}$ being the wave length of free waves of frequency σ. Assuming the ratio of actual amplitudes to the wavelength is small, one may express the normalized amplitudes for

the linear displacements using a small parameter α as $X^o = \alpha X^1$, $Y^o = \alpha Y^1$, and $Z^o = \alpha Z^1$. Similarly, the amplitudes for small angular displacements may be expressed as $\theta^o_x = \alpha\theta^1_x$, $\theta^o_y = \alpha\theta^1_y$, and $\theta^o_z = \alpha\theta^1_z$. Accordingly, neglecting the terms involving α^2, the velocity potential for the resulting fluid motion can be expressed as

$$\phi = \alpha\phi^1 \quad , \tag{2.7}$$

where ϕ^1 and its derivative are bounded. Furthermore, the linearized dynamic pressure is

$$\pi = -\rho\phi_t(\bar{x},\bar{y},\bar{z};t) = -\alpha\rho\phi^1_t\left(\frac{x}{k},\frac{y}{k},\frac{z}{k};t\right) \quad , \tag{2.8}$$

where ρ is the density of the fluid, and the linearized surface elevation becomes

$$\bar{\eta} \sim -\frac{1}{g}\,\phi_t(\bar{x},0,\bar{z};t) = -\frac{\alpha}{g}\,\phi^1_t\left(\frac{x}{k},0,\frac{z}{k};t\right) \quad . \tag{2.9}$$

Since the fluid is considered to be incompressible, the velocity potential must satisfy the Laplace equation,

$$\nabla^2\phi = 0 \quad ,$$

or

$$\nabla^2 U = 0 \qquad \text{in } \bar{y} < 0, \text{ outside } S \quad . \tag{2.10}$$

At a free surface, the first-order boundary condition to be satisfied is

$$\phi_{tt} + g\phi_{\bar{y}} = 0 \quad ,$$

or

$$U_{\bar{y}} - kU = 0 \qquad \text{on } \bar{y} = 0, \text{ outside } S \quad . \tag{2.11}$$

Next, on the immersed surface the kinematic condition can be written as

$$\phi_{\bar{n}} = \underline{n}(\bar{x},\bar{y},\bar{z}:t)\ \underline{\dot{r}}(\bar{x},\bar{y},\bar{z}:t) \quad \text{on } S(\bar{x},\bar{z}:t), \qquad (2.12)$$

where $\underline{n}$ represents the unit vector normal to the surface, i.e., $\underline{n} = in_x + jn_y + kn_z$, and $\underline{\dot{r}}$ is the velocity vector of a point on the surface. Note that the surface S of the ship in motion differs from the surface at rest S^o which was introduced earlier. Since

$$[\phi_{\bar{n}}]_{\bar{y}=S} = [\phi_{\bar{n}}]_{\bar{y}=S^o} + O(\alpha^2) \quad ,$$

it is consistent with the omission of the second order terms to require that condition (2.12) be satisfied on S^o. By substituting (2.5) into the right hand side of (2.12) one obtains

$$\phi_{\bar{n}} = \dot{\bar{X}}n_x + \dot{\bar{Y}}n_y + \dot{\bar{Z}}n_z + \dot{\theta}_x q_x + \dot{\theta}_y q_y + \dot{\theta}_z q_z$$

or

$$\sum_{j=1}^{6} [U_j(\bar{x},\bar{y},\bar{z})]_{\bar{n}} = -i\sigma[\bar{X}^o n_x + \bar{Y}^o n_y + \bar{Z}^o n_z + \theta_x^o q_x + \theta_y^o q_y + \theta_z^o q_z]$$

$$\text{on } S^o(\bar{x},\bar{z}) \quad . \qquad (2.13)$$

Since the problem is linearized, the total potential U is expressed as $\sum_{j=1}^{6} U_j$ $\quad j = 1, 2, --- 6$, and (q_x, q_y, q_z) represents the components of a vector given by $\underline{q} = \underline{r} \times \underline{n}$.

Finally, at large distances, the propagating disturbance must have the form of a radically outgoing wave, i.e.,

$$U(\bar{x},\bar{y},\bar{z}) - \frac{f(\theta)}{\sqrt{\bar{r}}} e^{k\bar{y}} e^{ik\bar{r}} = o\left(\frac{1}{\bar{r}}\right) \quad \text{as } \bar{r} \to \infty \quad , \qquad (2.14)$$

where $r = \sqrt{\bar{x}^2 + \bar{z}^2}$, and $\theta = \arctan\left(\frac{\bar{z}}{\bar{x}}\right)$.

To show clearly the dependence of the solution on parameters describing the type of oscillation, one relates the dynamic pressure functions $p_j(x,y,z)$ $j = 1, 2, \cdots 6$, to the potential function $U(\bar{x},\bar{y},\bar{z})$ by

$$g\left[\frac{X^o}{k} p_1 + \frac{Y^o}{k} p_2 + \frac{Z^o}{k} p_3 + \frac{a}{k} \theta^o_x p_4 + \frac{a}{k} \theta^o_y p_5 + \frac{a}{k} \theta^o_z p_6\right] = i\sigma \sum_{j=1}^{6} U_j\left(\frac{x}{k},\frac{y}{k},\frac{z}{k}\right) . \tag{2.15}$$

Then the boundary conditions (2.10), (2.11), (2.13), and (2.14) can be transformed into

$$\nabla^2 p_j = 0 \qquad \text{in } y < 0 , \quad \text{outside } S \tag{2.16}$$

$$(p_j)_y - p_j = 0 \qquad \text{on } y = 0 , \quad \text{outside } S \tag{2.17}$$

$$\left.\begin{aligned} &(p_1)_n = n_x , \quad (p_2)_n = n_y , \quad (p_3)_n = n_z \\ &(p_4)_n = \frac{y}{a} n_z - \frac{z}{a} n_y , \quad (p_5)_n = \frac{z}{a} n_x - \frac{x}{a} n_z , \\ &(p_6)_n = \frac{x}{a} n_y - \frac{y}{a} n_x \qquad \text{on } S^o(x,z) , \end{aligned}\right\} \tag{2.18}$$

$$p_j - \frac{f(\theta)}{\sqrt{r}} e^y e^{ir} = O\left(\frac{1}{r}\right) \qquad \text{as } r \to \infty . \tag{2.19}$$

where

$$n_x = \frac{-S_x}{\sqrt{1+(S_x)^2+(S_z)^2}} , \quad n_y = \frac{1}{\sqrt{1+(S_x)^2+(S_z)^2}} , \text{ and } n_z = \frac{-S_z}{\sqrt{1+(S_x)^2+(S_z)^2}} . \tag{2.20}$$

III. SHALLOW DRAFT APPROXIMATION

In this section, the first order forces and moments due to the dynamic pressure are related to the added mass, added moment of inertia, and damping factors of an oscillating ship. The perturbation procedure for a ship of shallow draft is developed.

The forces and moments due to the dynamic pressure exerted on the immersed surface are given by

$$\underline{F} = \iint_{S(\bar{x},\bar{z};t)} \pi(\bar{x},\bar{y},\bar{z};t)\cdot\underline{n}\, dS \quad , \tag{3.1}$$

and

$$\underline{G} = \iint_{S(\bar{x},\bar{z};t)} \pi(\bar{x},\bar{y},\bar{z};t)\cdot\underline{q}\, dS \quad . \tag{3.2}$$

Since, from (2.8), $\pi(\bar{x},\bar{y},\bar{z};t) = -\alpha\rho\phi^1_t$, and since $S(\bar{x},\bar{z};t) = S^o + \alpha S^1$, the first order forces and moments are

$$\underline{F}^1 = -\alpha\rho \iint_{S^o(\bar{x},\bar{z})} \phi^1_t(\bar{x},\bar{y},\bar{z};t)\cdot\underline{n}\, dS \quad , \tag{3.3}$$

and

$$\underline{G}^1 = -\alpha\rho \iint_{S^o(\bar{x},\bar{z})} \phi^1_t(\bar{x},\bar{y},\bar{z};t)\cdot\underline{q}\, dS \quad . \tag{3.4}$$

By (2.3) and (2.15), one finds

$$\phi^1_t = -g\,\mathrm{Re}[\underline{R}^1 p_j e^{-i\sigma t}], \text{ where } j = 1,2,3, \tag{3.5}$$

or

$$= -g\,\bar{a}\,\mathrm{Re}[\underline{\theta}^1 p_j e^{-i\sigma t}], \text{ where } j = 4,5,6. \tag{3.6}$$

Therefore, the substitutions of (3.5) into (3.3), and (3.6) into (3.4) yield

$$\underline{F}^1 = \alpha\rho g \iint_{S^o(\bar{x},\bar{z})} \mathrm{Re}[R^1 p_j e^{-i\sigma t}]\cdot\underline{n}\, dS \ , \quad \text{where } j = 1,2,3, \tag{3.7}$$

and

$$\underline{G}^1 = \alpha\rho g\bar{a} \iint_{S^o(\bar{x},\bar{z})} \mathrm{Re}[\underline{\theta}^1 p_j e^{-i\sigma t}]\cdot\underline{q}\, dS \ , \quad \text{where } j = 4,5,6. \tag{3.8}$$

Suppose the first order forces and moments are expressed in the form

$$\underline{F}^1 = -\underline{\widetilde{M}}\,\underline{\ddot{R}} - \underline{\widetilde{N}}\,\underline{\dot{R}} \ , \tag{3.9}$$

and

$$\underline{G}^1 = -\underline{\widetilde{I}}\,\bar{a}\,\underline{\ddot{\theta}} - \underline{\widetilde{H}}\,\bar{a}\,\underline{\dot{\theta}} \ , \tag{3.10}$$

where $\underline{\widetilde{M}}$ and $\underline{\widetilde{I}}$ are called the added mass, and added moment of inertia, while $\underline{\widetilde{N}}$ and $\underline{\widetilde{H}}$ are called the damping coefficients for translation and rotation, respectively.

By substituting (2.4) into (3.9) and (3.10) one obtains

$$\underline{F}^1 = \alpha\ \mathrm{Re}[(\sigma^2\underline{\widetilde{M}} + i\sigma\underline{\widetilde{N}})\underline{R}^1 e^{-i\sigma t}] \ , \tag{3.11}$$

and

$$\underline{G}^1 = \alpha\ \mathrm{Re}[(\sigma^2\underline{\widetilde{I}} + i\sigma\underline{\widetilde{H}})\bar{a}\ \underline{\theta}^1 e^{-i\sigma t}] \ . \tag{3.12}$$

In equating (3.7) to (3.11), and (3.8) to (3.12), all variables are made dimensionless by the transformation $x = k\bar{x}$, $y = k\bar{y}$, and $z = k\bar{z}$, then one has

$$k^3\tilde{\underline{M}}/\rho = \mathrm{Re}\left\{\iint_{S^o(x,z)} p_j \cdot \underline{n}\, dS\right\},\ k^3\tilde{\underline{N}}/\rho\sigma = \mathrm{Im}\left\{\iint_{S^o(x,z)} p_j \cdot \underline{n}\, dS\right\},\ \text{where } j = 1,2,3, \tag{3.13}$$

$$k^4\tilde{\underline{I}}/\rho = \mathrm{Re}\left\{\iint_{S^o(x,z)} p_j \cdot \underline{q}\, dS\right\},\ k^4\tilde{\underline{H}}/\rho\sigma = \mathrm{Im}\left\{\iint_{S^o(x,z)} p_j \cdot \underline{q}\, dS\right\},\ \text{where } j = 4,5,6. \tag{3.14}$$

$S^o(x,z)$ represents the image of $S^o(\bar{x},\bar{z})$, and hence $S^o(x,z) = k^2 S^o(\bar{x},\bar{z})$.

Next, it will be shown how the shallow draft approximation can be developed. Here the shallow draft means that the ratio of actual draft $\bar{\epsilon}$ to the wavelength $\bar{\lambda}$ is small. Therefore, one may begin by writing the hull of the ship in the form

$$y = \epsilon S^1(x,z) \qquad -a \le x \le a \qquad -b \le z \le b\ , \tag{3.15}$$

with $\epsilon = 2\pi\bar{\epsilon}/\bar{\lambda}$ as the small dimensionless parameter appropriate for the case of shallow draft. $S^1(x,z)$ satisfies the following conditions

$$S^1(0,0) = 1,\quad S^1(x,z) = S^1(-x,z),\ \text{and}\ S^1(x,z) = S^1(x,-z). \tag{3.16}$$

From (3.15), $[1+(S_x)^2+(S_z)^2]^{-\frac{1}{2}}$ can be expanded into

$$[1+(S_x)^2+(S_z)^2]^{-\frac{1}{2}} = 1 - \frac{\epsilon^2}{2}[(S_x^1)^2 + (S_z^1)^2]$$
$$+ \sum_{m=2}^{\infty} (-1)^m \epsilon^{2m} \alpha_m [(S_x^1)^2 + (S_z^1)^2]^m\ ,$$

thus one obtains

$$\frac{\partial}{\partial n} = -\frac{S_x}{\sqrt{1+(S_x)^2+(S_z)^2}}\frac{\partial}{\partial x} + \frac{1}{\sqrt{1+(S_x)^2+(S_z)^2}}\frac{\partial}{\partial y} - \frac{S_z}{\sqrt{1+(S_x)^2+(S_z)^2}}\frac{\partial}{\partial z}$$

$$= [-\epsilon S_x^1 + O(\epsilon^2)]\frac{\partial}{\partial x} + [1 + O(\epsilon^2)]\frac{\partial}{\partial y} + [-\epsilon S_z^1 + O(\epsilon^2)]\frac{\partial}{\partial z} \quad , \tag{3.17}$$

where $n_x = -\epsilon S_x^1 + O(\epsilon^2)$, $n_y = 1 + O(\epsilon^2)$, and $n_z = -\epsilon S_z^1 + O(\epsilon^2)$ (3.18)

Designating the right hand member of (2.18) as $R_j(x,z)$ $j = 1,2, --- 6$, these quantities are expanded into

$$R_j(x,z) = \sum_{m=0}^{\infty} \epsilon^m R_j^m(x,z) \quad . \tag{3.19}$$

From (3.18) one now finds

$$\left.\begin{aligned} &R_1^0(x,z) = 0,\ R_2^0(x,z) = 1,\ R_3^0(x,z) = 0,\ R_4^0(x,z) = -\frac{z}{a}, \\ &R_5^0(x,z) = 0,\ R_6^0(x,z) = \frac{x}{a},\ R_1^1(x,z) = -S_x^1,\ R_2^1(x,z) = 0, \\ &R_3^1(x,z) = -S_z^1,\ R_4^1(x,z) = 0,\ R_5^1(x,z) = -\frac{z}{a}S_x^1 + \frac{x}{a}S_z^1, \\ &R_6^1(x,z) = 0,\ ---\ . \end{aligned}\right\} \tag{3.20}$$

Similarly, the pressure would also have to be developed in the form

$$p_j(x,y,z) = \sum_{m=0}^{\infty} \epsilon^m p_j^m(x,y,z) \quad j = 1,2, --- 6 \quad . \tag{3.21}$$

Here one makes use of Taylor's theorem and writes p_j^m for $y = 0$ as

$$p_j^m[x,\epsilon S^1(x,z),z] = \sum_{n=0}^{\infty} \frac{1}{n!}[\epsilon S^1(x,z)]^n \frac{\partial^n}{\partial y^n} p_j^m(x,0,z). \qquad (3.22)$$

Substitution of expansions (3.17), (3.21) and (3.22) into the left hand member of (2.18) yield

$$[-\epsilon S_x^1+O(\epsilon^2)]\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n!}\epsilon^{m+n}[S^1(x,z)]^n \frac{\partial^{n+1}}{\partial x\partial y^n} p_j^m(x,0,z)$$

$$+ [1+O(\epsilon^2)]\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n!}\epsilon^{m+n}[S^1(x,z)]^n \frac{\partial^{n+1}}{\partial y^{n+1}} p_j^m(x,0,z) \qquad (3.23)$$

$$+ [-\epsilon S_z^1+O(\epsilon^2)]\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n!}\epsilon^{m+n}[S^1(x,z)]^n \frac{\partial^{n+1}}{\partial y^n\partial z} p_j^m(x,0,z).$$

Upon equating coefficients of like powers of ϵ in (3.19) and (3.23), one obtains

$$[p_j^o(x,0,z)]_y = R_j^o(x,z) \quad , \qquad (3.24)$$

$$[p_j^1(x,0,z)]_y = R_j^1(x,z) + S_x^1[p_j^o(x,0,z)]_x + S_z^1[p^o(x,0,z)]_z - S^1[p_j^o(x,0,z)]_{yy} \, ,$$

and so forth. Therefore, p_j^m can be determined recursively by solving boundary value problems of the following form.

Find a function $w(x,y,z)$ such that

$$\nabla^2 w = 0 \qquad \text{in } y < 0, \text{ outside } S \qquad (3.25)$$

$$w_y - w = 0 \qquad \text{on } y = 0, \text{ outside } S \qquad (3.26)$$

$$w_y = L(x,z) \qquad \text{on } y = 0, \text{ within } S \qquad (3.27)$$

$$w - \frac{f(\theta)}{\sqrt{r}} e^{ky} e^{ikr} = O(\frac{1}{r}) \qquad \text{as } r \to \infty \qquad (3.28)$$

Note that $h(x,z)$ is a function whose values are given by (3.20).

In evaluating the added mass, added moment of inertia, and damping coefficients, the half-length of the ship $\bar{a}$ appears as an integral limit. Therefore, all lengths will be made dimensionless by dividing them by $\bar{a}$ instead of the wavelength. This change can be readily carried out by introducing a new set of independent and dependent variables,

$$\tilde{x} = x/a \;, \qquad \tilde{y} = y/a \;, \qquad \tilde{z} = z/a \;,$$

and (3.29)

$$w(x,y,z) = aV(x/a,\ y/a,\ z/a) \quad \text{where } a = k\bar{a} \;.$$

The boundary value problems can be stated as follows:
Find a function $V(\tilde{x},\tilde{y},\tilde{z})$ such that

$$\nabla^2 V = 0 \qquad \text{in } \tilde{y} < 0 \;, \quad \text{outside } S \;, \tag{3.30}$$

$$V_{\tilde{y}} - aV = 0 \qquad \text{on } \tilde{y} = 0 \;, \quad \text{outside } S \;, \tag{3.31}$$

$$V_{\tilde{y}} = h(\tilde{x},\tilde{z}) \qquad \text{on } \tilde{y} = 0 \;, \quad \text{within } S \;, \tag{3.32}$$

$$V - \frac{f(\theta)}{\sqrt{\tilde{r}}}\, e^{a\tilde{y}} e^{ia\tilde{r}} = O\!\left(\frac{1}{\tilde{r}}\right) \qquad \text{as } \tilde{r} \to \infty \;. \tag{3.33}$$

Now the order of the added mass, added moment of inertia, and damping factors involving a specific type of oscillation will be examined by the use of expansions which resulted from the shallow draft approximation. Substitution of (3.18), (3.21) and (3.22) into (3.13) and (3.14) lead to the following expressions:

$$k^3\widetilde{M}_x/\rho = O(\epsilon), \qquad k^3\widetilde{N}_x/\rho\sigma = O(\epsilon),$$

$$k^3\widetilde{M}_y/\rho = \mathrm{Re}\Big\{\iint_{S^o} p_2^o(x,0,z)dS\Big\}, \qquad k^3\widetilde{N}_y/\rho\sigma = \mathrm{Im}\Big\{\iint_{S^o} p_2^o(x,0,z)dS\Big\},$$

$$k^3\widetilde{M}_z/\rho = O(\epsilon), \qquad k^3\widetilde{N}_z/\rho\sigma = O(\epsilon),$$

$$k^4\widetilde{I}_x/\rho = -\mathrm{Re}\Big\{\iint_{S^o} z\cdot p_4^o(x,0,z)dS\Big\}, \qquad k^4\widetilde{H}_x/\rho\sigma = -\mathrm{Im}\Big\{\iint_{S^o} z\cdot p_4^o(x,0,z)dS\Big\},$$

$$k^4\widetilde{I}_y/\rho = O(\epsilon), \qquad k^4\widetilde{H}_y/\rho\sigma = O(\epsilon),$$

$$k^4\widetilde{I}_z/\rho = \mathrm{Re}\Big\{\iint_{S^o} x\cdot p_6^o(x,0,z)dS\Big\}, \qquad k^4\widetilde{H}_z/\rho\sigma = \mathrm{Im}\Big\{\iint_{S^o} x\cdot p_6^o(x,0,z)dS\Big\}.$$

Hence, in terms of the new variables, one obtains the added mass and damping coefficient for heave as

$$\begin{aligned} M_y &= \widetilde{M}_y/\rho\bar{a}^3 = \mathrm{Re}\Big\{\iint_{\widetilde{S}} V_2(\widetilde{x},0,\widetilde{z})dS\Big\}, \\ N_y &= \widetilde{N}_y/\rho\bar{a}^3\sigma = \mathrm{Im}\Big\{\iint_{\widetilde{S}} V_2(\widetilde{x},0,\widetilde{z})dS\Big\}, \end{aligned} \tag{3.34}$$

the added moment of inertia and damping coefficient for roll as

$$\begin{aligned} I_x &= \widetilde{I}_x/\rho\bar{a}^4 = -\,\mathrm{Re}\Big\{\iint_{\widetilde{S}} \widetilde{z}\cdot V_4(\widetilde{x},0,\widetilde{z})dS\Big\}, \\ H_x &= \widetilde{H}_x/\rho\bar{a}^4\sigma = -\,\mathrm{Im}\Big\{\iint_{\widetilde{S}} \widetilde{z}\cdot V_4(\widetilde{x},0,\widetilde{z})dS\Big\}, \end{aligned} \tag{3.35}$$

and the added moment of inertia and damping coefficient for pitch as

$$\begin{aligned} I_z &= \widetilde{I}_z/\rho\bar{a}^4 = \mathrm{Re}\Big\{\iint_{\widetilde{S}} \widetilde{x}\cdot V_6(\widetilde{x},0,\widetilde{z})dS\Big\}, \\ H_z &= \widetilde{H}_z/\rho\bar{a}^4\sigma = \mathrm{Im}\Big\{\iint_{\widetilde{S}} \widetilde{x}\cdot V_6(\widetilde{x},0,\widetilde{z})dS\Big\}, \end{aligned} \tag{3.36}$$

where $\tilde{S}$ represents the image of $S^o(x,z)$ under the transformation $\tilde{x} = x/a$, and $\tilde{z} = z/a$, and which is equal to $\frac{1}{a^2} S^o(x,z)$. Furthermore, since $p_j^o(x,y,z) = aV_j(\tilde{x},\tilde{y},\tilde{z})$, $j = 2,4,6$, from (3.20) and (3.22) one finds

$$[V_2(\tilde{x},0,\tilde{z})]_{\tilde{y}} = 1, \ [V_4(\tilde{x},0,\tilde{z})]_{\tilde{y}} = \tilde{z}, \text{ and } [V_6(\tilde{x},0,\tilde{z})]_{\tilde{y}} = \tilde{x} \quad \text{on } \tilde{S} . \tag{3.37}$$

Also, from (3.21), one finds

$$[V_j(\tilde{x},0,\tilde{z})]_{\tilde{y}} - aV_j(\tilde{x},0,\tilde{z}) = 0 \quad \text{outside } \tilde{S} . \tag{3.38}$$

IV. INTEGRAL REPRESENTATION

An integral representation of the solution of the boundary value problem which is formulated in the preceding section will be discussed here. Consider a region (a fluid space) bounded by the hull of a ship and a free surface. An oscillating ship generates waves in this region which in turn react so as to influence the motion of the ship. A disturbance created at a point on the hull is transmitted to a point (x,y,z) some distance away through the fluid. As a result it will be shown that for some function f(x,z) defined over the hull, the resulting potential in the region may be expressed in the form,

$$V(x,y,z) = \frac{1}{4\pi} \iint_S f(\xi,\zeta) G(x,y,z,\xi,0,\zeta) d\xi d\zeta . \tag{4.1}$$

(Hereinafter, delete the tilde on variables for the sake of convenience.)

Physically, expression (4.1) may be interpreted as the field potential caused by distributed sources of the density $f(x,z)$ over the hull S, and then Green's function G represents the field potential caused by a unit point source at $(\xi,0,\zeta)$ on the hull. The appropriate Green's function [1] for the present problem is obtained by setting $\eta = 0$:

$$\begin{aligned} G(x,y,z,\xi,0,\zeta) &= \frac{1}{R} + \oint_0^\infty \frac{\beta+a}{\beta-a} e^{\beta y} J_o(\beta r) d\beta \ , \\ &= \frac{2}{R} + \oint_0^\infty \frac{2a}{\beta-a} e^{\beta y} J_o(\beta r) d\beta \ , \end{aligned} \tag{4.2}$$

$r^2 = (x-\xi)^2 + (z-\zeta)^2$, and $R^2 = r^2 + y^2$, where $J_o(\beta r)$ is the zero order Bessel function of the first kind, and the integral sign $\oint_0^\infty$ is to be understood as the integration along the positive real axis except for an arc in the lower half plane to avoid the positive real root $\beta = a$ of the denominator.

From (4.2) it can be shown that

$$G_y - aG = \frac{\partial}{\partial y} \frac{2}{R} \qquad \text{in } y < 0 \ . \tag{4.3}$$

If the integrals W and L are defined by

$$W(x,y,z) = \frac{1}{4\pi} \iint_S f(\xi,\zeta) \frac{2}{R} d\xi d\zeta \ , \tag{4.4}$$

and

$$L(x,y,z) = \frac{1}{4\pi} \iint_S f(\xi,\zeta) H(x,y,z) d\xi d\zeta \ , \tag{4.5}$$

where
$$H(x,y,z) = \oint_0^{\infty} \frac{2a}{\beta-a} J_o(\beta r)e^{\beta y}d\beta ,$$

it follows from (4.1) that $V(x,y,z) = W(x,y,z) + L(x,y,z)$. For a function $f(x,z)$ continuous on S, the integral L satisfies condition (3.30), and W + L satisfies condition (3.33). Furthermore, by (4.3) one finds

$$V_y(x,y,z) - aV(x,y,z) = W_y(x,y,z) \quad \text{in } y < 0. \quad (4.6)$$

It is a standard theorem of potential theory that W satisfies the relation

$$\begin{aligned} \lim_{\substack{y \to 0 \\ y<0}} W_y(x,y,z) &= f(x,z) && \text{within S ,} \\ &= 0 && \text{outside S .} \end{aligned} \quad (4.7)$$

Hence,

$$V_y(x,0,z) - aV(x,0,z) = f(x,z) \quad \text{within S ,} \quad (4.8)$$

and

$$V_y(x,0,z) - aV(x,0,z) = 0 \quad \text{outside S ,} \quad (4.9)$$

so that W + L also satisfies condition (3.31).

Now it can be seen from (3.32) that the potential $V(x,y,z)$ given by (4.1) will be the solution of the given boundary value problem if $f(x,z)$ is chosen as a solution of the following integral equation:

$$h(x,z) - aV(x,0,z) = f(x,z) \quad (4.10)$$

or

$$f(x,z) + \frac{a}{4\pi} \iint_S f(\xi,\zeta)G(x,0,z,\xi,0,\zeta)d\xi d\zeta = h(x,z) \quad \text{within S.} \qquad (4.11)$$

By making use of known integrals in [7], the kernel of the integral equation can be explicitly shown to be

$$G(x,0,z,\xi,0,\zeta) = \frac{2}{r} - \pi a[Y_o(ar) + S_o(ar) - i2J_o(ar)], \qquad (4.12)$$

where $Y_o(ar)$ denotes the zero order Bessel function of the second kind, and $S^o(ar)$ denotes the zero order Struve function, respectively. Here one observes that as ar becomes large, $S_o(ar) \approx Y_o(ar)$, hence Green's function (4.12) tends to

$$\begin{aligned} G(x,0,z,\xi,0,\zeta) &\sim \frac{2}{r} - 2\pi a[Y_o(ar) - iJ_o(ar)] \\ &\approx \frac{2}{r} - 2\pi a\left[\frac{\sin(ar-\pi/4)}{\sqrt{\pi ar/2}} - i\,\frac{\cos(ar-\pi/4)}{\sqrt{\pi ar/2}}\right] \\ &= \frac{2}{r} + i\,2\sqrt{\frac{2\pi a}{r}}\,e^{i(ar-\pi/4)} \quad . \end{aligned} \qquad (4.13)$$

Therefore, the kernel of the integral equation fluctuates as the frequency of the oscillation increases.

According to the Fredholm theory, Equation (4.11) is soluble if the homogeneous equation,

$$f^o(x,z) + \frac{a}{4\pi} \iint_S f^o(\xi,\zeta)G(x,0,z,\xi,0,\zeta)d\xi d\zeta = 0 \qquad (4.14)$$

possesses only the trivial solution, $f^o(x,z) = 0$. However, it has been shown in [1] that the potential constructed with the solution of Equation (4.14),

$$V^{o}(x,y,z) = \frac{1}{4\pi}\iint\limits_{S} f^{o}(\xi,\zeta)G(x,y,z,\xi,0,\zeta)d\xi d\zeta$$

vanishes identically in $y < 0$. Since V^{o} vanishes, one finds from (4.8) that $f^{o}(x,z)$ must also vanish. Hence, the integral Equation (4.11) can be solved.

Upon finding the solution $f(x,z)$, then from (3.37) and (4.10) the pressure may be determined by

$$p_2 = aV_2(x,0,z) = 1 - f_2(x,z), \quad \text{for heave} \tag{4.15}$$

$$p_4 = aV_4(x,0,z) = z - f_4(x,z), \quad \text{for roll} \tag{4.16}$$

$$p_6 = aV_6(x,0,z) = x - f_6(x,z), \quad \text{for pitch} \tag{4.17}$$

where $f(x,z) = \mathrm{Re}[f(x,z)] + \mathrm{Im}[f(x,z)]$. Then, substitution of (4.15) into (3.34) yields

$$\left.\begin{aligned} M_y &= \frac{1}{a}\iint\limits_{S} \left\{1 - \mathrm{Re}[f_2(x,z)]\right\} dS \quad , \\ N_y &= -\frac{1}{a}\iint\limits_{S} \mathrm{Im}[f_2(x,z)]\, dS \quad \text{for heave,} \end{aligned}\right\} \tag{4.18}$$

substitution of (4.16) into (3.35) yields

$$\left.\begin{aligned} I_x &= -\frac{1}{a}\iint\limits_{S} z\left\{z - \mathrm{Re}[f_4(x,z)]\right\} dS \quad , \\ H_x &= \frac{1}{a}\iint\limits_{S} z\, \mathrm{Im}[f_4(x,z)]\, dS \quad \text{for roll,} \end{aligned}\right\} \tag{4.19}$$

and substitution of (4.17) into (3.36) yields

$$\left.\begin{aligned} I_z &= \frac{1}{a}\iint\limits_S x\left\{x - \mathrm{Re}[f_6(x,z)]\right\} dS \quad , \\ H_z &= -\frac{1}{a}\iint\limits_S x\, \mathrm{Re}[f_6(x,z)]\, dS \qquad \text{for pitch,} \end{aligned}\right\} \tag{4.20}$$

V. NUMERICAL PROCEDURE

Suppose an ellipse $\left(\frac{\bar{x}}{\bar{a}}\right)^2 + \left(\frac{\bar{z}}{\bar{b}}\right)^2 = 1$, which becomes $x^2 + \left(\frac{z}{b}\right)^2 = 1$ under the transformation $x = \bar{x}/\bar{a}$, $z = \bar{z}/\bar{a}$, and $b = \bar{b}/\bar{a}$, represents the surface of a zero-draft ship. A numerical procedure will be presented which determines approximately the value of the unknown density $f(x,z)$ over the surface of the ship. The procedure is based on Fredholm's method of replacing the integral equation with a finite set of linear equations which relate the values of $f(x,z)$ at chosen pivotal points on the elliptic surface.

The substitution of (4.12) into (4.11) yields

$$2f(x,z) + \frac{a}{2\pi}\int_{-1}^{1}\int_{-b}^{b} f(\xi,\zeta)\left\{\frac{2}{r} - \pi a[Y_o(ar) + S_o(ar) - i2J_o(ar)\right\} d\zeta d\xi$$

$$= 2h(x,z) \qquad \text{on } y = S(x,z). \tag{5.1}$$

Since $f(x,z)$ is a complex function, (5.1) can be resolved into a pair of equations

$$\left.\begin{aligned}
&2\mathrm{Re}[f(x,z)] + \frac{a}{\pi}\int_{-1}^{1}\int_{-b}^{b}\mathrm{Re}[f(\xi,\zeta)]\left[\frac{1}{r} - a\log(ar) - \frac{\pi a}{2}R(ar)\right]d\zeta d\xi \\
&\qquad - a^2\int_{-1}^{1}\int_{-b}^{b}\mathrm{Im}[f(\xi,\zeta)]J_0(ar)\,d\zeta d\xi = 2h(x,z)\ , \\
&2\mathrm{Im}[f(x,z)] + a^2\int_{-1}^{1}\int_{-b}^{b}\mathrm{Re}[f(\xi,\zeta)]J_0(ar)\,d\zeta d\xi \\
&\qquad + \frac{a}{\pi}\int_{-1}^{1}\int_{-b}^{b}\mathrm{Im}[f(\xi,\zeta)]\left[\frac{1}{r} - a\log(ar) - \frac{\pi a}{2}R(ar)\right]d\zeta d\xi = 0,
\end{aligned}\right\}\quad(5.2)$$

where $R(ar) = Y_0(ar) + S_0(ar) - \frac{2}{\pi}\log(ar)$. $R(ar)$ is a regular function at $r = 0$.

A lattice is established on the elliptic surface by dividing the long axis into sixteen equal intervals h, and the vertical ordinates parallel to the short axis into eight equal intervals $k(x)$ [see Figure 1]. Then each pivotal point can be identified by the coordinates

$$\left.\begin{aligned}
x_i &= (i-8)h && j = 0,1, \text{---}\ 16, \text{ where } h = 1/8\ , \\
z_j(x_i) &= (j-4)k(x_i) && j = 0,1, \text{---}\ 8, \text{ where } k(x_i) = \tfrac{b}{4}\sqrt{1-x_i^2}.
\end{aligned}\right\}(5.3)$$

To obtain the solution $f(x_i,z_j)$, it is necessary to consider Equations (5.2) only at the forty-one pivotal points contained in one quadrant. (The function becomes either symmetric or anti-symmetric relative to the coordinate axes depending upon the type of oscillation.)

Since r denotes the distance from a given point P (x_i,z_j), where $i = 0,1,$ --- 8, and $j = 0,1,$ --- 4, to a variable point Q $(\xi,\zeta) = (x_l,z_m)$, where $l = 0,1,$ --- 16, and $m = 0, 1,$ --- 8, one finds the

integrands

$$f(\xi,\zeta)/\sqrt{(x_i-\xi)^2+(z_j-\zeta)^2}, \quad \text{and} \quad f(\xi,\zeta)\log a\sqrt{(x_i-\xi)^2+(z_j-\zeta)^2}$$

possess a singularity at $(x_i,z_j) = (x_i,z_m)$. Nevertheless, these improper integrals associated with the above integrands do exist. A numerical scheme will be shown for evaluating such integrals involving two variables.

A. Treatment of the Integral.

$$I_S(x_i,z_j) = \int_{x_o}^{x_{16}}\int_{z_o}^{z_8} \frac{f(\xi,\zeta)}{\sqrt{(x_i-\xi)^2+(z_j-\zeta)^2}}\, d\zeta d\xi \ . \qquad (5.4)$$

In the given lattice, division lines and curves form a four-sided (two or three sided near the boundary) mesh. A neighboring region δ of a point P is defined as a region which is the union of all meshes touching P. Then the region S of the integration may be divided into two parts, so that

$$I_S(x_i,z_j) = I_\delta(x_i,z_j) + I_{S-\delta}(x_i,z_j) \quad ,$$

where, for example, if δ is a neighboring region about an interior point P,

$$I_\delta(x_i,z_j) = \int_{x_{i-1}}^{x_{i+1}}\int_{z_{j-1}}^{z_{j+1}} \frac{f(\xi,\zeta)}{\sqrt{(x_i-\xi)^2+(z_j-\zeta)^2}}\, d\zeta d\xi \ . \quad (5.5)$$

For the evaluation of $I_{S-\delta}$, Simpson's rule is generally used. In case there are an odd number of divisions in an interval of integration, the integral over the division nearest to the region δ is evaluated by the trapezoidal rule instead.

Now, setting $\xi - x_i = p$, and $\zeta - z_j = q$, (5.5) may be expressed as

$$I_\delta(x_i,z_j) = \int_{-h}^{h}\int_{-k}^{k} \frac{g(p,q)}{\sqrt{p^2+q^2}}\,dq\,dp = \int_{-k}^{k} F(h,q)\,dq \quad , \qquad (5.6)$$

where $g(p,q) = f(x_i+p,\ z_j+q)$, and $F(h,q) = \int_{-h}^{h} \frac{g(p,q)}{\sqrt{p^2+q^2}}\,dp$. Since $f(x,z)$ is continuous and bounded, it can be shown that $F(h,q)$ is also a bounded and continuous function of q for $q \neq 0$. When q approaches zero, $F(h,q)$ becomes logarithmically singular. Therefore, a generalized Gaussian quadrature formula is used for an approximate evaluation of I_δ. To determine this formula, rewrite (5.6) in the form

$$I_\delta(x_i,z_j) = \int_{-k}^{k} \log|q| \left[\frac{F(h,q)}{\log|q|}\right] dq \approx \sum_{i=1}^{2} H_i F^*(h,q_i) \ . \qquad (5.7)$$

The abscissae q_i can be found as the zeroes of the second degree polynomial $Q(q) = 1 + aq + bq^2$, which is a member of the set of polynomials orthogonal in the interval $(-k,k)$ for a weight function $\log|q|$. It turns out that

$$Q(q) = 1 - \left(\frac{3}{k}\right)^2 \frac{1 - \log k}{1 - 3\log k}\, q^2 \quad ,$$

so that

$$q_1 = -\frac{k}{3}\sqrt{\frac{1-3\log k}{1-\log k}}, \text{ and } q_2 = \frac{k}{3}\sqrt{\frac{1-3\log k}{1-\log k}} \;. \tag{5.8}$$

Now the weights H_i corresponding to q_i can be determined by requiring that formula (5.7) be exact if $F^*(h,q)$ is a polynomial of degree zero or one. In the present case these are

$$H_1 = H_2 = k(\log k - 1) \;. \tag{5.9}$$

By substituting (5.8) and (5.9) into (5.7) one obtains a quadrature formula

$$I_\delta(x_i,z_j) \approx k(\log k-1)\left[\int_{-h}^{h} \frac{g(p,q_1)}{\log q_2\sqrt{p^2+q_2^2}}\,dp + \int_{-h}^{h} \frac{g(p,q_2)}{\log q_2\sqrt{p^2+q_2^2}}\,dp\right]$$

$$= k(\log k-1)\left[\int_{x_{i-1}}^{x_{i+1}} \frac{f(\zeta,z_j+q_1)}{\log q_2\sqrt{(x_i-\zeta)^2+q_2^2}}\,d\zeta \right.$$

$$\left. + \int_{x_{i-1}}^{x_{i+1}} \frac{f(\zeta,z_j+q_2)}{\log q_2\sqrt{(x_1-\zeta)^2+q_2^2}}\,d\zeta\right] \tag{5.10}$$

which gives exact results if $F^*(h,q)$ is a polynomial of degree less than or equal to three.

Next, by Lagrange's interpolation formula, $f(x,z_j+q_1)$ and $f(x,z_j+q_2)$ can be related to the function evaluated at the equally spaced points $f(x,z_{j-1})$, $f(x,z_j)$ and $f(x,z_{j+1})$. Thus, one finally finds the formula

$$I_\delta(x_i,z_j) \approx \frac{\log k - 1}{k \log q_2} \int_{x_{i-1}}^{x_{i+1}} \Big[q_2^2 f(\xi, z_{j-1}) + 2(k_2^2 - q_2^2) f(\xi, z_j) + q_2^2 f(\xi, z_{j+1}) \Big] \frac{1}{\sqrt{(x_i-\xi)^2 + q_2^2}}\, d\xi \,. \tag{5.11}$$

The integral I_δ expressed by (5.11) is now readily evaluated by the use of Simpson's rule.

For the boundary points (x_i, z_o) $i = 1, 2, \text{---}\ 8$, the associated I_δ is treated in a quite similar manner. Here one writes

$$I_\delta(x_i,z_o) = \int_{x_{i-1}}^{x_{i+1}} \int_{z_o}^{z_1} \frac{f(\xi,\zeta)}{\sqrt{(x_i-\xi)^2 + (z_o-\zeta)^2}}\, d\zeta\, d\xi \quad . \tag{5.12}$$

Following the transformations (5.6) and (5.7), (5.12) can be expressed in the form

$$I_\delta(x_i,z_o) = \int_{-h}^{h} \int_{0}^{k} \frac{g(p,q)}{\sqrt{p^2+q^2}}\, dq\, dp = \int_0^k \log|q| \left[\frac{F(h,q)}{\log|q|} \right] dq \approx \sum_{i=1}^{2} H_i F^*(h,q_i). \tag{5.13}$$

Then the abscissae q_i can be found from the quadratic equation

$$\left[\int_0^k \log q\, dq \int_0^k q^2 \log q\, dq - \left(\int_0^k q \log q\, dq \right)^2 \right] q^2 + \int_0^k q \log q\, dq \left[\int_0^k q^2 \log q\, dq - \int_0^k q^3 \log q\, dq \right] q + \int_0^k q \log q\, dq \int_0^k q^3 \log q\, dq - \left(\int_0^k q^2 \log q\, dq \right)^2 = 0 \,. \tag{5.14}$$

Note that q_i depends upon $k(x_i)$ in a rather complicated manner. Here, the weights H_i corresponding to q_i are given by

$$\left.\begin{aligned} H_1 &= \frac{1}{q_2-q_1}\left[q_2 \int_0^k \log q\,dq - \int_0^k q \log q\,dq\right], \\ H_2 &= \frac{-1}{q_2-q_1}\left[q_1 \int_0^k \log q\,dq - \int_0^k q \log q\,dq\right]. \end{aligned}\right\} \qquad (5.15)$$

Hence, the quadrature formula involving the functions evaluated at the equally spaced points $f(x,z_o)$ and $f(x,z_1)$ becomes

$$I_\delta(x_i,z_o) \approx \int_{x_{i-1}}^{x_{i+1}} \left[\frac{H_1(k-q_1)}{k \log q_1} \frac{1}{\sqrt{(x_i-\zeta)^2+q_1^2}} + \frac{H_2(k-q_2)}{k \log q_2} \frac{1}{\sqrt{(x_i-\zeta)^2+q_2^2}}\right] f(\zeta,z_o)d\zeta$$

$$+ \int_{x_{i-1}}^{x_{i+1}} \left[\frac{H_1 q_1}{k \log q_1} \frac{1}{\sqrt{(x_i-\zeta)^2+q_1^2}} + \frac{H_2 q_2}{k \log q_2} \frac{1}{\sqrt{(x_i-\zeta)^2+q_2^2}}\right] f(\zeta,z_1)d\zeta. \quad (5.16)$$

The integral I_δ associated with the point $(x_o z_4)$ at the leading edge of the ellipse also requires a special treatment. From the definition of the neighboring region, one starts by writing

$$I_\delta(x_o,z_4) = \int_{z_o}^{z_8}\int_{x_o}^{x_1} \frac{f(\zeta,\zeta)}{\sqrt{(x_o-\zeta)^2+(z_4-\zeta)^2}}\,d\zeta d\zeta = \int_0^{8k}\int_0^{h} \frac{g(p,q)}{\sqrt{p^2+q^2}}\,dpdq$$

$$= \int_0^h \log|p|\left[\frac{F(p,k)}{\log|p|}\right]dp \approx \sum_{i=1}^{2} L_i F^*(p,k) \,. \qquad (5.17)$$

Then the abscissae p_i can be found from (5.14) by replacing the variable q with p, and the integral limit k with h, respectively. It also follows

from (5.15) that the weights L_i corresponding to p_i are given by

$$\left.\begin{aligned} L_1 &= \frac{1}{p_2-p_1}\left[p_2\int_0^h \log p\,dp - \int_0^h p\log p\,dp\right], \\ L_2 &= \frac{-1}{p_2-p_1}\left[p_1\int_0^h \log p\,dp - \int_0^h p\log p\,dp\right]. \end{aligned}\right\} \tag{5.18}$$

The quadrature formula can now be written as

$$I_\delta(x_o,z_4) \approx \int_{z_o(p_1)}^{z_8(p_1)} \frac{L_1(h-p_1)}{h\log p_1}\frac{f(x_o,\zeta)}{\sqrt{p_1^2+(z_4-\zeta)^2}}d\zeta + \int_{z_o(p_2)}^{z_8(p_2)} \frac{L_2(h-p_2)}{h\log p_2}\frac{f(x_o,\zeta)}{\sqrt{p_2^2+(z_4-\zeta)^2}}d\zeta$$

$$+ \int_{z_o(p_1)}^{z_8(p_1)} \frac{L_1p_1}{h\log p_1}\frac{f(x_1,\zeta)}{\sqrt{p_1^2+(z_4-\zeta)^2}}d\zeta + \int_{z_o(p_2)}^{z_8(p_2)} \frac{L_2p_2}{h\log p_2}\frac{f(x_1,\zeta)}{\sqrt{p_2^2+(z_4-\zeta)^2}}d\zeta.$$

(5.19)

Since $z_4 = 0$, and since $f[x_o,z_j(p_i)] = f(x_o,z_4)$ $j = 0,1,$ --- $8,$ (5.19) may be rewritten as

$$I_\delta(x_o,z_4) \approx \left[\int_{z_o(p_1)}^{z_8(p_1)} \frac{L_1(h-p_1)}{h\log p_1}\frac{1}{\sqrt{p_1^2+\zeta^2}}d\zeta + \int_{z_o(p_2)}^{z_8(p_2)} \frac{L_2(h-p_2)}{h\log p_2}\frac{1}{\sqrt{p_2^2+\zeta^2}}d\zeta\right] f(x_o,z_4)$$

$$+ \int_{z_o(p_1)}^{z_8(p_1)} \frac{L_1p_1}{h\log p_1}\frac{f(x_1,\zeta)}{\sqrt{p_1^2+\zeta^2}}d\zeta + \int_{z_o(p_2)}^{z_8(p_2)} \frac{L_2p_2}{h\log p_2}\frac{f(x_1,\zeta)}{\sqrt{p_2^2+\zeta^2}}d\zeta. \tag{5.20}$$

Here, functions $f[x_1,z_j(p_i)]$ may be approximated by $f[x_1,z_j(x_1)]$ (or related to $f[x_1,z_j(x_1)]$ by linear interpolation) to evaluate the third and fourth integrals.

B. Treatment of the Integral.

$$J_S(x_i,z_j) = \int_{x_0}^{x_{16}} \int_{z_0}^{z_8} f(\xi,\zeta)\log\sqrt{(x_i-\xi)^2+(z_j-\zeta)^2}\, d\zeta d\xi \; . \qquad (5.21)$$

If the region S of integration is divided into a neighboring region δ and a remaining region, one obtains

$$J_S(x_i,z_j) = J_\delta(x_i,z_j) + J_{S-\delta}(x_i,z_j) \quad .$$

Applying Simpson's and the trapezoidal rules, the integral $J_{S-\delta}$ can be evaluated by a process similar to the one used for the evaluation of $I_{S-\delta}$. For example, if δ is a neighboring region about interior points P, the integral J_δ is given by

$$J_\delta(x_i,z_j) = \int_{x_{i-1}}^{x_{i+1}} \int_{z_{j-1}}^{z_{j+1}} f(\xi,\zeta)\log\sqrt{(x_i-\xi)^2+(z_j-\zeta)^2}\, d\zeta d\xi \; . \qquad (5.22)$$

By the transformation $\xi-x_i = p$, $\zeta-z_j = q$, (5.22) may be rewritten as

$$J_\delta(x_i,z_j) = \int_{-h}^{h} \int_{-k}^{k} g(p,q)\log\sqrt{p^2+q^2}\, dpdq = \int_{-k}^{k} G(h,q)dq \; , \qquad (5.23)$$

where $g(p,q) = f(x_i+p,\ z_j+q)$, and $G(h,q) = \int_{-h}^{h} g(p,q)\log\sqrt{p^2+q^2}\, dp$. It can be shown that $G(h,q)$ is not only a bounded and continuous function of q for $q \neq 0$, but also remains finite when q approaches zero. Therefore, non-generalized Gaussian quadrature formula may be used to evaluate J_δ. In order to derive such a formula, write (5.23) in the form

$$J_\delta(x_i,z_j) = \int_{-k}^{k} G(h,q)dq \approx \sum_{i=1}^{2} M_i G(h,q_i) \quad . \tag{5.24}$$

The abscissae q_i are obtained as zeroes of the second degree polynomial $Q(q) = 1 - \frac{3}{k^2} q^2$; thus $q_1 = -\frac{k}{\sqrt{3}}$, and $q_2 = \frac{k}{\sqrt{3}}$. (5.25)

Now, the weights M_i corresponding to q_i are

$$M_1 = M_2 = H \quad . \tag{5.26}$$

Then, substitution of (5.25) and (5.26) yields a quadrature formula

$$\begin{aligned} J_\delta(x_i,z_j) &\approx k\Big[\int_{-h}^{h} g(p,q_1)\log\sqrt{p^2+q_2^2}\,dp + \int_{-h}^{h} g(p,q_2)\log\sqrt{p^2+q_2^2}\,dp\Big] \\ &= k\Big[\int_{x_{i-1}}^{x_{i+1}} f(\zeta,z_j+q_1)\log\sqrt{(x_i-\zeta)^2+q_2^2}\,d\zeta + \int_{x_{i-1}}^{x_{i+1}} f(\zeta,z_j+q_2)\log\sqrt{(x_i-\zeta)^2+q_2^2}\,d\zeta\Big] \end{aligned} \tag{5.27}$$

which again gives exact results if $G(h,q)$ is a polynomial of degree less than or equal to three.

If $f(x,z_j+q_1)$ and $f(x,z_j+q_2)$ are expressed in terms of $f(x,z_{j-1})$, $f(x,z_j)$ and $f(x,z_{j+1})$ by Lagrange's interpolation formula, one finally finds

$$\begin{aligned} J_\delta(x_i,z_j) &\approx \frac{1}{k}\int_{x_{i-1}}^{x_{i+1}} \Big[q_2^2 f(\zeta,z_{j-1}) + 2(k^2-q_2^2)f(\zeta,z_j) + q_2^2 f(\zeta,z_{j+1})\Big]\log\sqrt{(x_i-\zeta)^2+q_2^2}\,d\zeta \\ &= \frac{k}{3}\int_{x_{i-1}}^{x_{i+1}} \Big[f(\zeta,z_{j-1}) + 4f(\zeta,z_j) + f(\zeta,z_{j+1})\Big]\log\sqrt{(x_i-\zeta)^2+q_2^2}\,d\zeta \quad . \end{aligned} \tag{5.28}$$

Note that quadrature coefficients are equal to those of Simpson's rule. For boundary points (x_j, z_o) $i = 1,2, --- 8$, the integral J_δ is

$$J_\delta(x_i,z_o) = \int_{x_{i-1}}^{x_{i+1}} \int_{z_o}^{z_1} f(\xi,\zeta)\log\sqrt{(x_i-\xi)^2+(z_o-\zeta)^2}\, d\zeta d\xi \; . \qquad (5.29)$$

Then (5.29) can be transformed into

$$J_\delta(x_i,z_o) = \int_{-h}^{h} \int_{0}^{k} g(p,q)\ln\sqrt{p^2+q^2}\, dqdp = \int_0^k G(h,q) \approx \sum_{i=1}^{2} M_i G(h,q_i). \quad (5.30)$$

In the present case, the abscissae q_i and corresponding M_i are readily found as

$$\left.\begin{aligned} q_1 &= \frac{k}{2}\left(1-\frac{1}{\sqrt{3}}\right), \quad q_2 = \frac{k}{2}\left(1+\frac{1}{\sqrt{3}}\right), \\ &\text{and} \\ H_1 &= H_2 = k/2 \; . \end{aligned}\right\} \qquad (5.31)$$

Hence, in terms of $f(x,z_o)$ and $f(x,z_1)$, the quadrature formula (5.30) can be explicitly shown as

$$J_\delta(x_i,z_o) \approx \int_{x_{i-1}}^{x_{i+1}} \left[\frac{k-q_1}{2}\log\sqrt{(x_i-\xi)^2+q_1^2} + \frac{k-q_2}{2}\log\sqrt{(x_i-\xi)^2+q_2^2}\right] f(\xi,z_o)d\xi$$

$$+ \int_{x_{i-1}}^{x_{i+1}} \left[\frac{q_1}{2}\log\sqrt{(x_i-\xi)^2+q_1^2} + \frac{q_2}{2}\log\sqrt{(x_i-\xi)^2+q_2^2}\right] f(\xi,z_1)d\xi \; . \qquad (5.32)$$

Note that quadrature coefficients are similar to those of the trapezoidal rule. The integral J_δ associated with the boundary point (x_o,z_4) requires a special treatment. Now, a quadrature formula will be derived by writing

$$J_\delta(x_o,z_4) = \int_{z_o}^{z_8}\int_{x_o}^{x_1} f(\xi,\zeta)\log\sqrt{(x_o-\xi)^2+(z_4-\zeta)^2}\,d\xi d\zeta = \int_0^{8k}\int_0^h g(p,q)\log\sqrt{p^2+q^2}\,dpdq$$

$$= \int_0^h G(p,k)dp \approx \sum_{i=1}^{2} N_i G(p_i,k)\ . \qquad (5.33)$$

Then, from (5.31) one finds

$$\left.\begin{aligned} p_1 &= \frac{h}{2}\left(1-\frac{1}{\sqrt{3}}\right), \quad p_2 = \frac{h}{2}\left(1+\frac{1}{\sqrt{3}}\right), \\ &\text{and} \\ N_1 &= N_2 = h/2\ . \end{aligned}\right\} \qquad (5.34)$$

Since $f[x_o,z_j(p_i)] = f(x_o,z_4)$ $j = 0,1,$ --- 8, one obtains the quadrature formula

$$J_\delta(x_o,z_4) \approx \left[\int_{z_o(p_1)}^{z_8(p_1)} \frac{h-p_1}{2}\log\sqrt{p_1^2+\zeta^2}\,d\zeta + \int_{z_o(p_2)}^{z_8(p_2)} \frac{h-p_2}{2}\log\sqrt{p_2^2+\zeta^2}\,d\zeta\right] f(x_o,z_4)$$

$$+ \int_{z_o(p_1)}^{z_8(p_1)} \frac{p_1}{2}\log\sqrt{p_1^2+\zeta^2}\, f(x_1,\zeta)d\zeta + \int_{z_8(p_2)}^{z_8(p_2)} \frac{p_2}{2}\log\sqrt{p_2^2+\zeta^2}\, f(x_1,\zeta)d\zeta\ .$$

$$(5.35)$$

Here, functions $f[x_1,z_j(p_i)]$ may be approximated by $f[x_1,2_j(x_1)]$ (or related to $f[x_1,z_j(x_1)]$ by linear interpolation.)

Application of the singular integral formulae:

for inner points	(5.11) and (5.28),
for boundary points	(5.16) and (5.32),
for leading-edge points	(5.20) and (5.35),

together with Simpson's and the trapezoidal rules enable one to reduce

the pair of integral equations (5.2) to eighty-two linear equations relating the values of $\mathrm{Re}[f(x,z)]$ and $\mathrm{Im}[f(x,z)]$ at forty-one pivotal points.

For example, if (x_i, z_j) are inner points, (5.2) is reduced to the form

$$\begin{aligned}
2\mathrm{Re}[f^{ij}] + \frac{a}{\pi}\Bigg\{ & \sum_{\ell=i-1}^{i+1} \sum_{m=j-1}^{j+1} \left(S'_{\ell m} \frac{1}{r'_{\ell m}} - aS''_{\ell m} \log ar''_{\ell m} \right) + \left[\sum_{\ell=0}^{i-1} \sum_{m=0}^{8} + \sum_{\ell=i+1}^{16} \sum_{m=0}^{8} \right. \\
& \left. + \sum_{\ell=i-1}^{i+1} \sum_{m=0}^{j-1} + \sum_{\ell=i-1}^{i+1} \sum_{m=j+1}^{8} \right] \overline{K}_{\ell m} \left(\frac{1}{r_{\ell m}} - a \log ar_{\ell m} \right) \\
& - \sum_{\ell=0}^{16} \sum_{m=0}^{8} \overline{K}_{\ell m} \frac{\pi a}{2} R(ar_{\ell m}) \Bigg\} \mathrm{Re}[f^{\ell m}] - a^2 \sum_{\ell=0}^{16} \sum_{m=0}^{8} \overline{K}_{\ell m} J_o(ar_{\ell m}) \mathrm{Im}[f^{\ell m}] = 2h^{ij},
\end{aligned}$$

$$\begin{aligned}
a^2 \sum_{\ell=0}^{16} \sum_{m=0}^{8} \overline{K}_{\ell m} J_o(ar_{\ell m}) \mathrm{Re}[f^{ij}] + 2\mathrm{Im}[f^{ij}] + \frac{a}{\pi}\Bigg\{ & \sum_{\ell=i-1}^{i+1} \sum_{m=j-1}^{j+1} \left(S'_{\ell m} \frac{1}{r'_{\ell m}} - aS''_{\ell m} \log ar''_{\ell m} \right) \\
& + \left[\sum_{\ell=0}^{i-1} \sum_{m=0}^{8} + \sum_{\ell=i+1}^{16} \sum_{m=0}^{8} + \sum_{\ell=i-1}^{i+1} \sum_{m=0}^{j-1} + \sum_{\ell=i-1}^{i+1} \sum_{m=j+1}^{8} \right] \overline{K}_{\ell m} \left(\frac{1}{r_{\ell m}} - a \log ar_{\ell m} \right) \\
& - \sum_{\ell=0}^{16} \sum_{m=0}^{8} \overline{K}_{\ell m} \frac{\pi a}{2} R(ar_{\ell m}) \Bigg\} \mathrm{Im}[f^{\ell m}] = 0 \qquad (5.36)
\end{aligned}$$

where the notations represent, respectively,

$$\begin{aligned}
& f^{ij} = f(x_i, z_j), \quad f^{\ell m} = f(x_\ell, z_m), \quad r_{\ell m} = \sqrt{(x_i - x_\ell)^2 + (z_j - z_m)^2} \\
& r'_{\ell m} = \sqrt{(x_i - x_\ell)^2 + \frac{k^2}{9} \frac{(1 - 3 \log k)}{(1 - \log k)}}, \quad r''_{\ell m} = \sqrt{(x_i - x_\ell)^2 + \frac{k^2}{3}}, \quad h^{ij} = h(x_i, z_j),
\end{aligned}$$

$$\left.\begin{array}{l} \bar{K}_{\ell m} = C_\ell h\, D_m k(x_\ell) \quad \text{(The coefficients for Simpson's or the trapezoidal rule)}, \\ S'_{\ell m} = C_\ell h \dfrac{q_2^2}{k} \dfrac{\log k-1}{\log q_2} \text{ or } C_\ell h \dfrac{2(k^2-q_2^2)}{k} \dfrac{\log k-1}{\log q_2},\; S''_{\ell m} = C_\ell h\, D_m k(x_\ell). \end{array}\right\} \tag{5.37}$$

In (5.36), an appropriate change is necessary if (x_i, z_j) are boundary points. Furthermore, one observes that the function $f(x,z)$ has the following properties;

$$\left.\begin{array}{ll} \text{for heave} & f^{\ell,m} = f^{17-\ell,m} = f^{\ell,9-m} = f^{17-\ell,9-m}\,, \\ \text{for roll} & f^{\ell,m} = f^{17-\ell,m} = -f^{\ell,9-m} = -f^{17-\ell,9-m} \\ \text{for pitch} & f^{\ell,m} = -f^{17-\ell,m} = f^{\ell,9-m} = -f^{17-\ell,9-m} \end{array}\right\} \tag{5.38}$$

Therefore, using (5.38) the left hand side of (5.36) can be formed into a rectangular matrix for unknowns f^{ij}. In this manner, one obtains three sets of linear equations describing the type of forced oscillation. For various values of the frequency parameter $a = \bar{a}\sigma^2/g = 2\pi\bar{a}/\bar{\lambda}$, one can determine the values of the density function by solving these equations.

Actual steps of the computation work for obtaining the solution consist of:

[Step 1] Determination of the distance r from the given pivotal point to all pivotal points on the lattice.

[Step 2] Calculation of functions appearing in the coefficients of the linear equations by means of interpolation from the given table.

[Step 3] Summation of the coefficients and grouping of the matrix in accordance with the type of oscillation.

[Step 4] Numerical solution of linear equations by the elimination process based on the algorithm of Gauss.

VI. NUMERICAL RESULTS

The results presented in this paper are obtained by the application of a new numerical scheme developed for evaluating improper double integrals associated with the fundamental singularities $1/r$ and $\log r$. As it will be shown, the present results agree fairly well with the results computed by the application of the previous scheme (see Section V in [8]) which uses a circular neighboring region about the singularity. Secondly, in the present work, the added moment of inertia and damping coefficient for roll and pitch are computed correctly. These quantities presented in [8] are in error. It was discovered that a mistake was committed in Step 3 of solving linear equations for the source density. In the course of grouping the matrix in accordance with rolling or pitching oscillation, the signs occuring in (5.38) were unwittingly reversed. Subsequently, the added moment of inertia and damping coefficient were determined from erroneous values of the pressure. Lastly, in the present work, the range of the computation was extended up to the frequency parameter $a = 2\pi$. However, as has been seen in (4.13) because of the fluctuating tendency of the kernel at large values of the argument, the accuracy of results decreases beyond $a = \pi$. To assure the same accuracy over

the range of $a = \pi$ to 2π, the mesh spacing may be bisected.

The investigation was performed for an elongated ellipse of axes ratio $b = \bar{b}/\bar{a} = 1/8$, and $1/4$, and its limiting case, a circle of $b = 1$. Thus, three groups of computations were necessary with each value of the frequency parameter a, which assume the values $\pi/6$, $\pi/5$, $\pi/4$, $\pi/3$, $2\pi/5$, $\pi/2$, $2\pi/3$, π, $5\pi/4$, $3\pi/2$, $7\pi/4$, and 2π.

For each combination of b and a, three sets of linear equations describing the type of oscillations; namely, heave, roll and pitch, were solved. From the solutions of the linear equations, pivotal values of the pressure were determined. Then, using Simpson's rule one finds normalized added mass M, normalized added moment of inertia I, and normalized damping coefficients N or H by the pair of appropriate formulas (4.18) to (4.20).

In order to substantiate the preceding remark, the values of M_y and N_y computed by the use of the previous scheme and those computed by use of the present scheme are compared in Table 1.

TABLE 1

	Previous Result				Present Result			
	b=1		b=1/8		b=1		b=1/8	
a	M_y	N_y	M_y	N_y	M_y	N_y	M_y	N_y
$\pi/5$	2.129	1.016	0.102	0.032	2.118	1.022	0.113	0.031
$\pi/4$	1.987	1.026	0.098	0.036	1.981	1.031	0.108	0.034
$\pi/3$	1.809	1.002	0.090	0.041	1.807	1.006	0.099	0.038
$2\pi/5$	1.706	0.968	0.085	0.043	1.704	0.970	0.093	0.040
$\pi/2$	1.593	0.910	0.077	0.045	1.592	0.909	0.084	0.041
$2\pi/3$	1.473	0.811	0.066	0.044	1.467	0.807	0.072	0.040
π	1.341	0.628	0.051	0.039	1.319	0.638	0.056	0.033

In Figure 2 and Figure 3, the dependence of $\frac{\pi\bar{a}^2}{A} M_y$ and $\frac{\pi\bar{a}^2}{A} N_y$ on the parameter a are presented. In Figure 4 and Figure 5, the quantities $\frac{\pi\bar{a}^2}{A} I_x$ and $\frac{\pi\bar{a}^2}{A} H_x$ are plotted as functions of the parameter a. Similarly, the quantities $\frac{\pi\bar{a}^2}{A} I_z$ and $\frac{\pi\bar{a}^2}{A} H_z$ are plotted in Figure 6 and Figure 7, respectively. The multiplication factor $\frac{\pi\bar{a}^2}{A} = \frac{\bar{a}}{\bar{b}}$ for the ordinates represents the ratio of the area of a circle with radius equal to $\bar{a}$ (half the long axis of the disk) to the area of the waterplane of the disk under consideration. This factor was introduced in order to make the curves comparable. The actual values correspond to one-fourth of the ordinate for the ellipse of b = 1/4, and one-eighth of the ordinate for the ellipse of b = 1/8, respectively.

The curves for the circular disk b = 1 in Figure 2 and Figure 3 compare very closely to the corresponding curves in Figure 6 and Figure 7 of [5], which were obtained by treating the circular disk problem by deleting the angular dependence. The same method is applied to solve the case of pitching in [6]. In Table 2, the results obtained by the present method and those obtained by the method of [6] are compared. In the latter method, the accuracy of the computation is increased by means of doubling the numbers of pivotal points on the radius of the circular disk. Over the range of a = π to 2π, the present method gives the lower values of I_z and H_z which amounts to 3 to 14% for I_z and 8 to 20% for H_z, respectively.

TABLE 2

For a Pitching Circular Disk

	By Present Method		By Method of [6]	
a	I_z	H_z	I_z	H_z
π/6	0.281	0.013	0.282	0.014
π/5	0.285	0.020	0.287	0.021
π/4	0.289	0.031	0.293	0.034
π/3	0.289	0.051	0.294	0.055
2π/5	0.284	0.064	0.290	0.070
π/2	0.273	0.079	0.279	0.085
2π/3	0.252	0.091	0.259	0.097
π	0.219	0.091	0.227	0.099
5π/4	0.203	0.086	0.212	0.095
3π/2	0.190	0.079	0.203	0.091
7π/4	0.176	0.071	0.198	0.085
2π	0.167	0.063	0.194	0.079

It should be noted that as the frequency of the forced oscillation σ tends to zero (hence, $a \to 0$), M_y becomes a constant and N_y (being a S^2) the damping factor will vanish. Since M_y is O(log a), in the two-dimensional case, M_y tends to infinity while N_y becomes a constant as the frequency tends to zero. In Figure 3, the limiting values of the ordinate may be shown by the relation

$$\lim_{a \to 0} \frac{\pi \bar{a}^2}{A} N_y = \frac{\pi \bar{a}^2}{A} (a\ S^2) = \frac{\bar{b}}{\bar{a}} \pi^2 a$$

where $A = \pi \bar{a} \bar{b}$, and $S = \pi \bar{a} \bar{b} / \bar{a}^2 = \pi \bar{b} / \bar{a}$ for the elliptic disk. Hence, at the origin, the slopes are given by

for b = 1 $\quad \pi^2 = 9.8696$,

for b = 1/4 $\quad 1/4\pi^2 = 2.4674$,

for b = 1/8 $\quad 1/8\pi^2 = 1.2337$, respectively.

The computed results deviate very rapidly from the low frequency approximation shown in the above. Furthermore, as the frequency tends to zero in Figure 4 and Figure 6, I_x and I_z become constants, while in Figure 5 and Figure 7, H_x and H_z will vanish.

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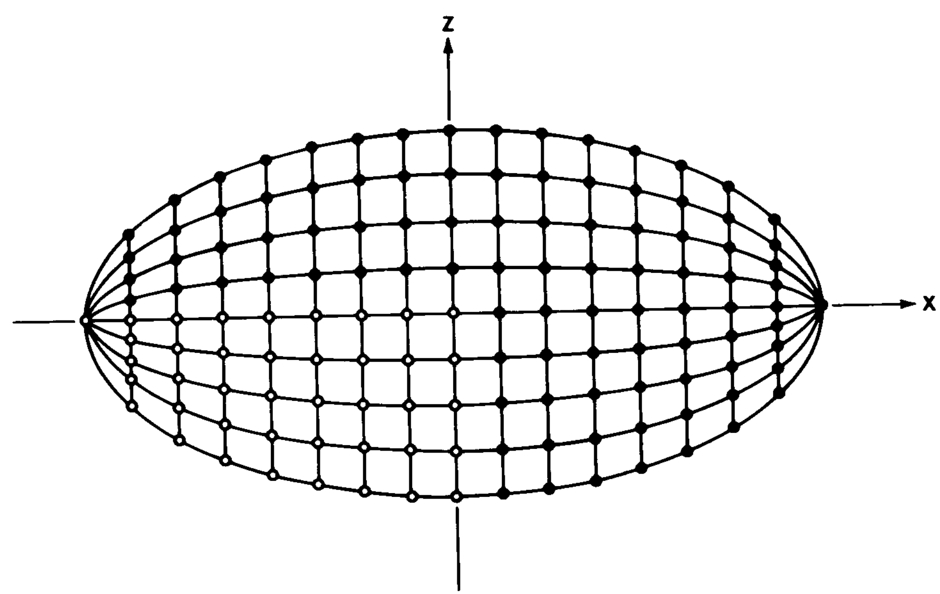


FIGURE 1

FIGURE 2

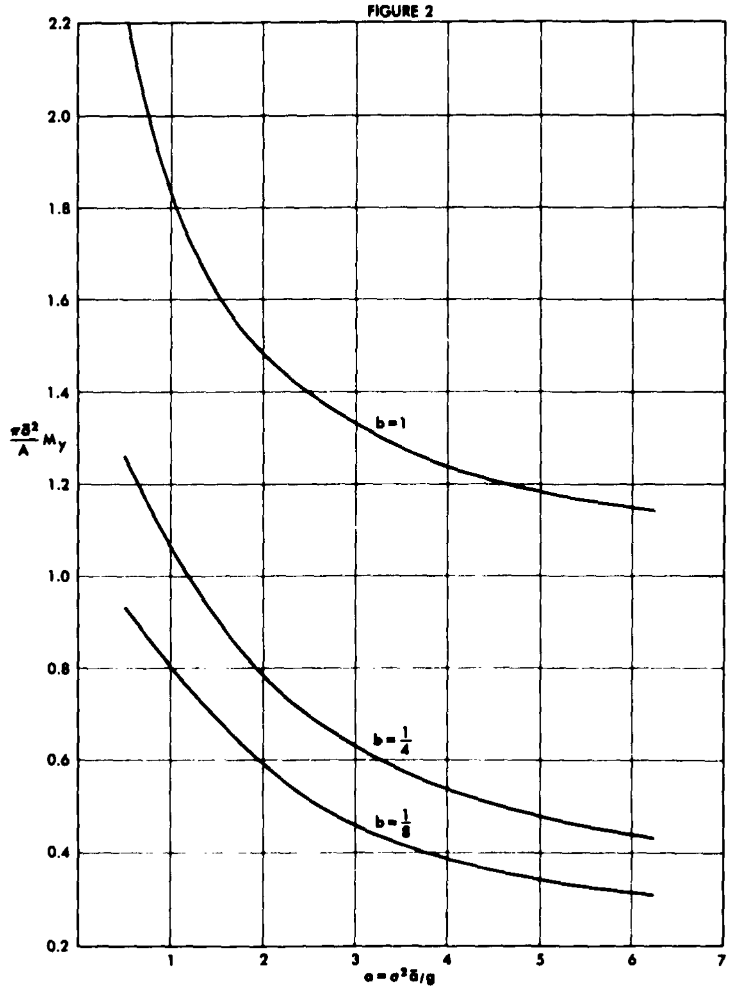

2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
$\frac{\pi\delta^2}{A} M_y$
b = 1
b = 1/4
b = 1/8
1
2
3
4
5
6
7
$a = \sigma^2 \delta / g$

FIGURE 3

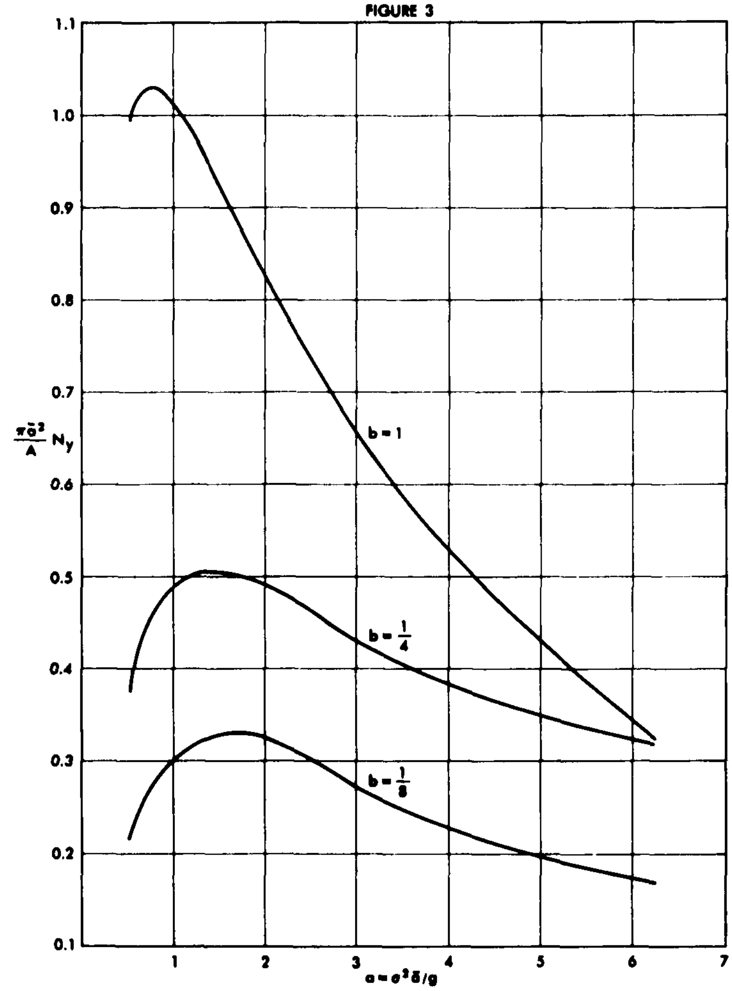
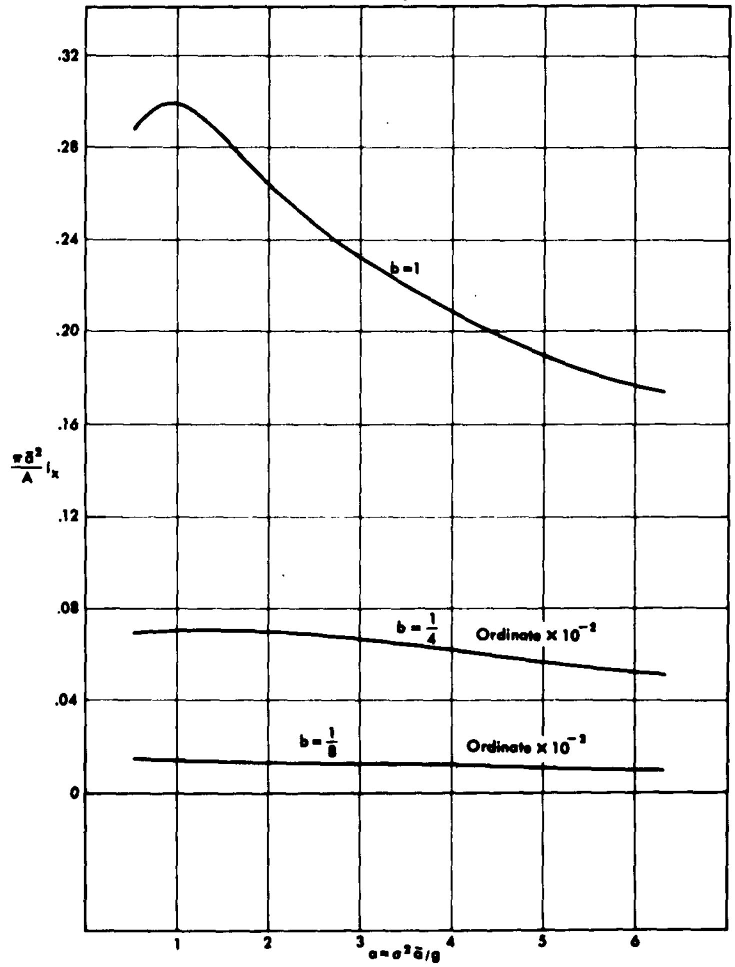
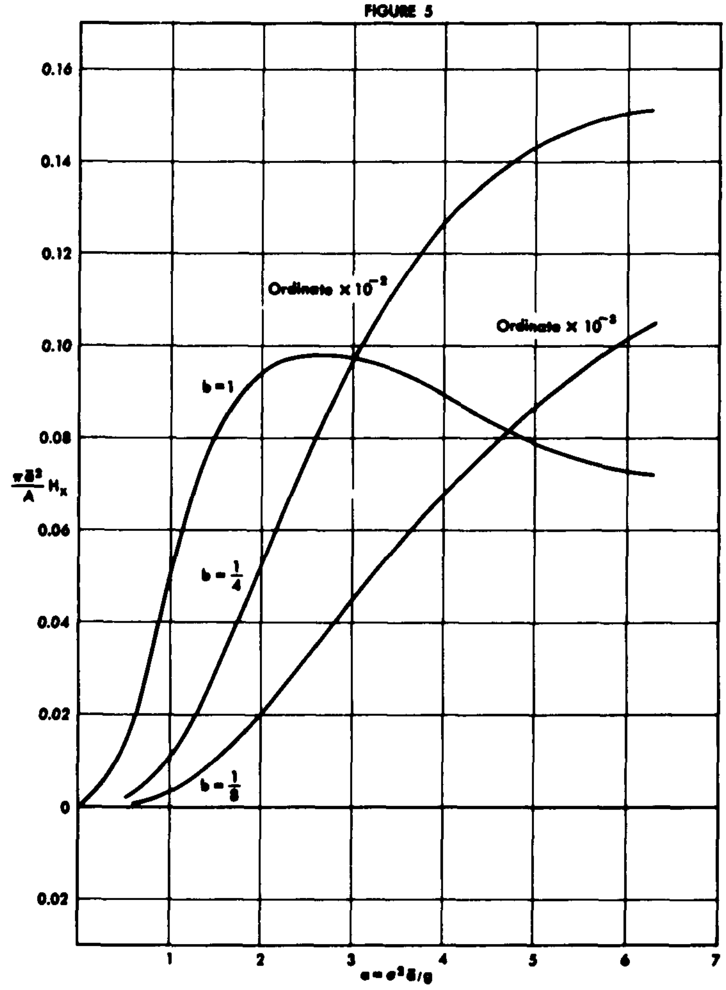


FIGURE 4



.32
.28
.24
.20
.16
.12
.08
.04
0
$\frac{\pi \bar{\delta}^2}{A} i_x$
b = 1
$b = \frac{1}{4}$
Ordinate × 10^{-2}
$b = \frac{1}{8}$
Ordinate × 10^{-3}
1
2
3
4
5
6
$a = \sigma^2 \bar{a}/g$

FIGURE 5



0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
0.02
$\frac{\pi \bar{\delta}^2}{A} H_x$
Ordinate $\times 10^{-2}$
Ordinate $\times 10^{-2}$
b = 1
b = $\frac{1}{4}$
b = $\frac{1}{8}$
1
2
3
4
5
6
7
$a = \sigma^2 \bar{\delta}/g$

FIGURE 6

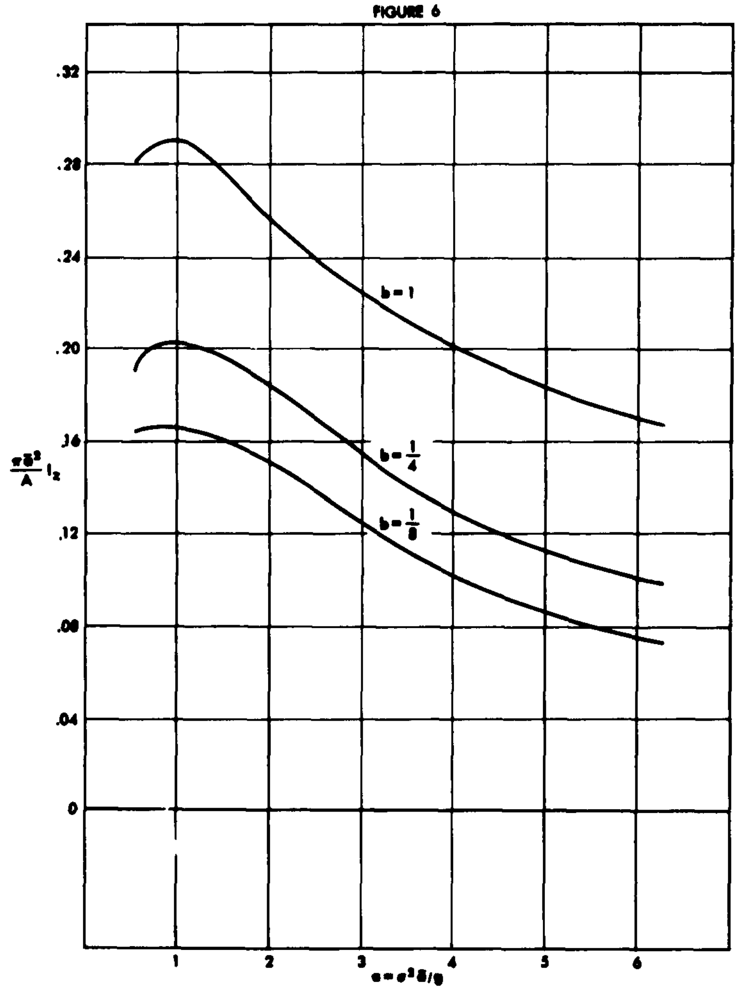


FIGURE 7